Combinatorial optimization tolerances calculated in linear time

Goldengorin, Boris; Sierksma, Gerard

Combinatorial Optimization Tolerances Calculated in Linear Time

Boris Goldengorin and Gerard Sierksma
Faculty of Economic Sciences, University of Groningen,
P.O. Box 800, 9700 AV Groningen, Netherlands
b.goldengorin,g.sierksma@eco.rug.nl

July 23, 2003

Abstract

For a given optimal solution to a combinatorial optimization problem, we show, under very natural conditions, the equality of the minimal values of upper and lower tolerances, where the upper tolerances are calculated for the given optimal solution and the lower tolerances outside the optimal solution. As a consequence, the calculation of such tolerances can now be done in linear time, while all current methods use quadratic time.

1 Introduction; Libura's Theorem Generalized

A *combinatorial optimization problem* $\mathrm{COP}(\mathcal{E}, C, \mathcal{S}, f_C)$ is the problem of finding

$$S^* \in \arg OPT\{f_C(S) \mid S \in \mathcal{S}\},$$

where $C : \mathcal{E} \to \Re$ is the given *instance* of the problem with a *ground set* $\mathcal{E}$ satisfying $|\mathcal{E}| = m$ $(m \geq 1)$, $\mathcal{S} \subseteq 2^{\mathcal{E}}$ is the *set of feasible solutions*, and $f_C : 2^{\mathcal{E}} \to \Re$ is the *objective function* of the problem. By $\mathcal{S}^* = \arg OPT\{f_C(S) \mid S \in \mathcal{S}\}$ the set of optimal solutions is denoted. It is assumed that $\mathcal{S}^* \neq \emptyset$, and that $S \neq \emptyset$ for some $S \in \mathcal{S}$. Let $g \in \mathcal{E}$, and $\alpha \geq 0$. By $C_{\alpha,g} : \mathcal{E} \to \Re$ we denote the instance defined as $C_{\alpha,g}(e) = C(e)$ for each $e \in \mathcal{E} \setminus \{g\}$, and $C_{\alpha,g}(g) = C(g) + \alpha$. Take any $S^* \in \mathcal{S}^*$. The *upper tolerance*, $u_{S^*}(e)$, of e with respect to S^* is defined as

$$u_{S^*}(e) = \max\{\alpha \geq 0 : S^* \in \arg\min\{f_{C_{\alpha,e}}(S) : S \in \mathcal{S}\}\},$$

and the *lower tolerance*, $l_{S^*}(e)$, with respect to S^* as

$$l_{S^*}(e) = \max\{\alpha \geq 0 \,:\, S^* \in \arg\min\{f_{C_{-\alpha,e}}(S) \,:\, S \in \mathcal{S}\}\}.$$

I.e. $u_{S^*}(e)$ is the maximal increase of $C(e)$ under which S^* stays optimal, and $l_{S^*}(e)$ is the maximal decrease of $C(e)$ under which S^* stays optimal. In Section 2 it will be shown that under very natural conditions, it holds that $\min\{u_{S^*}(e) \,:\, e \in S^*\} = \min\{l_{S^*}(e) \,:\, e \in \mathcal{E} \setminus S^*\}$. For an extensive account on properties of upper and lower tolerances in the context of sensitivity analysis, see among others, Gal [1] and Greenberg [2].

The following theorem can be seen as a generalization of Libura's well known theorem on tolerances (see, Libura [3], and Ramasvamy&Chakravarti [4]). We will use the following extra notations. Let $e \in \mathcal{E}$. Then $\mathcal{S}_+(e) = \{S \in \mathcal{S} \,:\, e \in S\}$, and $\mathcal{S}_-(e) = \{S \in \mathcal{S} \,:\, e \notin S\}$. Clearly, $\mathcal{S} = \mathcal{S}_-(e) \cup \mathcal{S}_+(e)$ and $\mathcal{S}_-(e) \cap \mathcal{S}_+(e) = \emptyset$ for all $e \in \mathcal{E}$. Similarly, $\mathcal{S}^*_+(e)$ and $\mathcal{S}^*_-(e)$ are the sets of optimal solutions containing e, and not containing e, respectively.

Theorem 1 *Consider a COP $(\mathcal{E}, C, \mathcal{S}, f_C)$. For each $S^* \in \mathcal{S}^*$ the following holds:*

1. $e \in \cap\mathcal{S}^*$ *iff* $u_{S^*}(e) = f_C(S) - f_C(S^*) > 0$ *for each* $S \in \mathcal{S}^*_-(e)$, $l_{S^*}(e) = \infty$;
2. $e \in \mathcal{E} \setminus \cup\mathcal{S}^*$ *iff* $u_{S^*}(e) = \infty$, $l_{S^*}(e) = f_C(S) - f_C(S^*) > 0$ *for each* $S \in \mathcal{S}^*_-(e)$;
3. $e \in S^* \setminus \cap\mathcal{S}^*$ *iff* $u_{S^*}(e) = 0$, $l_{S^*}(e) = \infty$;
4. $e \in \cup\mathcal{S}^* \setminus S^*$ *iff* $u_{S^*}(e) = \infty$, $l_{S^*}(e) = 0$.

The proof of Theorem 1 is left to the reader. Note that if $|\mathcal{S}^*| = 1$, then this theorem boils down to Libura's theorem on tolerances. Also note that if $\mathcal{S}_-(e) = \emptyset$ for some $e \in \mathcal{E}$, then $u_{S^*}(e) = \min\{f_C(T) \,:\, T \in \mathcal{S}_-(e)\} - f_C(S^*) = \min\{\emptyset\} = \infty$ (by definition). Similarly, for $\mathcal{S}_+(e) = \emptyset$ we take $l_{S^*}(e) = \infty$.

2 Minimal Upper and Lower Tolerances

We call $\mathcal{S}^*$ *nested w.r.t.* S^* for $S^* \in \mathcal{S}^*$ if and only if either $\cup\mathcal{S}^* = S^*$ or $\cap\mathcal{S}^* = S^*$; otherwise – *nonnested w.r.t.* S^*. We start with the rather trivial case of multi-optimal solutions.

Theorem 2 *Consider a COP$(\mathcal{E}, C, \mathcal{S}, f_C)$ with $|\mathcal{S}^*| \geq 2$. Then for any $S^* \in \mathcal{S}^*$ the following holds.*

1. *For $\mathcal{S}^*$ nested w.r.t. S^*:*

a. $S^* = \cap\mathcal{S}^*$ *implies that* $\min\{l_{S^*}(e) : e \in \mathcal{E} \setminus S^*\} = 0$, $\min\{u_{S^*}(e) : e \in S^*\} > 0$, *and* $< \infty$ *iff* $\mathcal{S}_-(e) \neq \emptyset$ *for some* $e \in S^*$*;*

b. $S^* = \cup\mathcal{S}^*$ *implies that* $\min\{u_{S^*}(e) : e \in S^*\} = 0$, $\min\{l_{S^*}(e) : e \in \mathcal{E} \setminus S^*\} > 0$, *and* $< \infty$ *iff* $\mathcal{S}_+(e) \neq \emptyset$ *for some* $e \in \mathcal{E} \setminus S^*$*;*

2. For $\mathcal{S}^*$ *nonnested w.r.t.* S^**:* $\min\{u_{S^*}(e) : e \in S^*\} = 0 = \min\{l_{S^*}(e) : e \in \mathcal{E} \setminus S^*\}$.

Proof. 1a. Since $|\mathcal{S}^*| \geq 2$, it follows that $\cup\mathcal{S}^* \setminus S^* \neq \emptyset$. Hence, $l_{S^*}(e) = 0$ for each $e \in \cup\mathcal{S}^* \setminus S^*$ (see Theorem 1(4)). Therefore, $\min\{l_{S^*}(e) : e \in \mathcal{E} \setminus S^*\} = 0$. Moreover, since $S^* \subseteq T$ for all $T \in \mathcal{S}^*$, it follows that $u_{S^*}(e) > 0$ for each $e \in S^*$, while $u_{S^*}(e) < \infty$ iff $S_-(e) \neq \emptyset$ for some $e \in S^*$ (see Theorem 1(1)).
1b. The proof is similar to the proof of 1a. Note that now $\cup(\mathcal{S}^* \setminus \{S^*\}) \neq \emptyset$.
2. $S^* \setminus \cap\mathcal{S}^* \neq \emptyset$ implies that $\min\{u_{S^*}(e) : e \in S^*\} = \min\{u_{S^*}(e) : e \in S^* \setminus \cap\mathcal{S}^*\} = 0$, and $\cup\mathcal{S}^* \setminus S^* \neq \emptyset$ implies that $\min\{l_{S^*}(e) : e \in \mathcal{E} \setminus S^*\} = \min\{l_{S^*}(e) : e \in \cup\mathcal{S}^* \setminus S^*\} = 0$. ■

Since the situation for $|\mathcal{S}^*| = 1$ is somehow different from the situation with $|\mathcal{S}^*| \geq 2$, we have devoted a special theorem for the case with a unique optimal solution. Below we write $u_{S^*} = u$ and $l_{S^*} = l$. Moreover, we use the set

$$\mathcal{S}^2 = \{S \in \mathcal{S} : S \in \arg\min\{f_C(S) : S \in \mathcal{S} \setminus \mathcal{S}^*\}\},$$

i.e., the set of second-best solutions of COP$(\mathcal{E}, C, \mathcal{S}, f_C)$ with $\mathcal{S}^* = \{S^*\}$ (see e.g. Van der Poort *et al.* [6]). We also assume that f_C is *monotone*, meaning that for each $S_1, S_2 \in 2^{\mathcal{E}}$, it holds that

$$\text{if} \quad S_1 \subseteq S_2 \quad \text{then} \quad f_C(S_1) \leq f_C(S_2).$$

Sum functions with $f_C(S) = \sum_{e \in S} C(e)$, *bottleneck functions* with $f_C(S) = \max_{e \in S} C(e)$, and *product functions* with $f_C(S) = \prod_{e \in S} C(e)$ and $C(e) \geq 1$ for each $e \in \mathcal{E}$ are all monotone functions.

Theorem 3 *Consider COP*$(\mathcal{E}, C, \mathcal{S}, f_C)$ *with monotone* f_C*, and unique optimal solution* S^**. Then the following holds.*

1. $\min\{u(e) : e \in S^*\} = \min\{l(e) : e \in \mathcal{E} \setminus S^*\} = \infty$ *iff* $\mathcal{S}_-(e) = \emptyset$ *for all* $e \in S^*$ *and* $\mathcal{S}_+(e) = \emptyset$ *for all* $e \in \mathcal{E} \setminus S^*$*;*

2. $0 < \min\{u(e) : e \in S^*\} < \min\{l(e) : e \in \mathcal{E} \setminus S^*\} = \infty$ *iff* $\mathcal{S}_-(e) \neq \emptyset$ *for all* $e \in S^*$ *and* $\mathcal{S}_+(e) = \emptyset$ *for all* $e \in \mathcal{E} \setminus S^*$*;*

3. $0 < \min\{l(e) : e \in \mathcal{E} \setminus S^*\} < \min\{u(e) : e \in S^*\} = \infty$ *iff* $\mathcal{S}_-(e) = \emptyset$ *for all* $e \in S^*$ *and* $\mathcal{S}_+(e) \neq \emptyset$ *for all* $e \in \mathcal{E} \setminus S^*$*;*

4. $0 < \min\{l(e) : e \in \mathcal{E} \setminus S^*\} \leq \min\{u(e) : e \in S^*\} < \infty$ *iff* $\mathcal{S}_-(e) \neq \emptyset$ *for all* $e \in S^*$ *and* $\mathcal{S}_+(e) \neq \emptyset$ *for all* $e \in \mathcal{E} \setminus S^*$*;*
moreover, $\mathcal{S}^2$ *is nonnested w.r.t.* S^* *iff* $0 < \min\{u(e) : e \in S^*\} = \min\{l(e) : e \in \mathcal{E} \setminus S^*\} < \infty$.

Proof. 1. This is by definition.
2. If $\mathcal{S}_-(e) \neq \emptyset$ for some $e \in S^*$, then there is a set $S \in \mathcal{S}_-(e)$ with $f_C(S) - f_C(S^*) > 0$. Hence, $\min\{u(e) : e \in S^*\} = \min\{\min\{f_C(S) : S \in \mathcal{S}_-(e)\} : e \in S^*\} - f_C(S^*) > 0$.
3. The proof of this part is similar to the proof of (2).
Since $\mathcal{S}_-(e) \neq \emptyset$ for some $e \in S^*$, there exists an element $e_0 \in S^*$ and a set $S_0 \in \mathcal{S}_-(e)$ such that $u(e_0) = \min\{u(e) : e \in S^*\} = f_C(S_0) - f_C(S^*)$. Clearly $S_0 \setminus S^* \neq \emptyset$, because otherwise $S_0 \subseteq S^*$ and the fact that f_C is monotone would imply that $f_C(S_0) = f_C(S^*)$, contradicting the fact that S^* is unique. Now let $t \in S_0 \setminus S^* \subseteq \mathcal{E} \setminus S^*$. Then $S_0 \in \mathcal{S}_+(t)$. Hence,

$$\min\{l(e) : e \in \mathcal{E} \setminus S^*\} \leq l(t) = \min\{f_C(S) : S \in \mathcal{S}_+(t)\} - f_C(S^*) = u(e_0).$$

The "moreover" part can be seen as follows. Since $\mathcal{S}^2$ is nonnested w.r.t. S^*, there is a set $S^0 \in \mathcal{S}^2$ such that $S^0 \setminus S^* \neq \emptyset \neq S^* \setminus S^0$. Then for each $e^0 \in S^0 \setminus S^*$ and each $e \in S^* \setminus S^0$, it follows that:

$$\begin{aligned}
&\min\{u(e) : e \in S^*\} \leq u(e) = \min\{f_C(T) : T \in \mathcal{S}_-(e)\} - f_C(S^*) = \\
&f_C(S^0) - f_C(S^*) = \min\{f_C(T) : T \in \mathcal{S}_+(e^0)\} - f_C(S^*) = \\
&\min\{l(e) : e \in \mathcal{E} \setminus S^*\}.
\end{aligned}$$

If $\mathcal{S}^2$ is nested w.r.t. S^*, then $\min\{l(e) : e \in \mathcal{E} \setminus S^*\} = l(e)$ for each $e \in S \setminus S^*$ with $S \in \mathcal{S}^2$, while $\min\{u(e) : e \in S^*\} = f_C(T) - f_C(S^*)$ for some T with $f_C(T) > f_C(S)$ for $S \in \mathcal{S}^2$. This completes the "iff" of the "moreover" part of 4. ∎

Theorem 4 *Consider COP*$(\mathcal{E}, C, \mathcal{S}, f_C)$ *with monotone* f_C*. If for each* $S_1, S_2 \in \mathcal{S}$ *with* $S_1 \neq S_2$*, it holds that neither* $S_1 \subset S_2$ *nor* $S_2 \subset S_1$*, then*

$$\min\{u_{S^*}(e) : e \in S^*\} = \min\{l_{S^*}(e) : e \in \mathcal{E} \setminus S^*\},$$

for each $S^* \in \mathcal{S}^*$*.*

Proof. If $|\mathcal{S}^*| \geq 2$, then the assertion of this theorem follows from Theorem 2(2). If $|\mathcal{S}^*| = 1$ the assertion follows from Theorem 3(4), because both $\mathcal{S}^*$ and $\mathcal{S}^2$ are nonnested w.r.t. S^* for each $S^* \in \mathcal{S}^*$. ∎

Note that the conditions from Theorem 4 hold for "regular" COPs, such as the traveling salesman, quadratic assignment, linear ordering, assignment, max-flow, shortest path, minimum spanning tree, and matching problems. The more irregular situation of Theorem 3(4) with " $>$ ", occurs in plant location problems. The following example shows such a situation with sum function f_C.

Let $\mathcal{E} = \{e_1, e_2, e_3\}$ with $C(e_1) = 1$, $C(e_2) = 2$, $C(e_3) = 4$, $S_1 = \{e_1\}$, $S_2 = \{e_1, e_2\}$, and $S_3 = \{e_3\}$. Then $\mathcal{S}^* = \{S_1\}$, $\mathcal{S}^2 = \{S_2\}$ with $u(e_1) = f_C(S_3) - f_C(S_1) = 3 > l(e_2) = f_C(S_2) - f_C(S_1) = 2$. Note that $l(e_3) = f_C(S_3) - f_C(S_1) = 3$.

The above example may suggest that there exist "easy" conditions for which

$$\max\{u_{S^*}(e) \,:\, e \in S^*\} = \max\{l_{S^*}(e) \,:\, e \in \mathcal{E} \setminus S^*\} \quad \text{for} \;\; S^* \in \mathcal{S}^*.$$

However, the following example points in the opposite direction. Let

$$C = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 0 \end{bmatrix}$$

be the cost matrix of an assignment problem with three jobs and three machines. One can easily check that $S^* = \{(1,1), (2,2), (3,3)\}$, and that $\max\{u(e) \,:\, e \in S^*\} = u(3,3) = 4 \neq 6 = l(3,1) = \max\{l(e) \,:\, e \in \mathcal{E} \setminus S^*\}$.

Finally, note that Theorem 3, and more specifically Theorem 4, allow us to compute the minimal upper and lower tolerances w.r.t. a given optimal solution S^*, within S^* and outside S^* respectively, by solving at most $\min\{O(|S^*|), O(|\mathcal{E} \setminus S^*|)\}$ (sub)COPs. This can be seen as a considerable improvement, in comparison to the situation where only Theorem 1 (including Libura's theorem) is available. In the latter case, $O(|\mathcal{E}|)$ (sub)COPs need to be solved; see also Van Hoesel&Wagelmans [5], and Ramasvamy&Chakravarti [4].

Acknowledgment. Authors are very thankful to Gregory Gutin and Marek Libura for their comments on the early draft of this paper.

References

[1] T. Gal. Postoptimal Analysis, Parametric Programming and Related Topics. McGraw-Hill, New York, 1979.

[2] H. Greenberg. An annotated bibliography for post-solution analysis in mixed integer and combinatorial optimization. In: D. L. Woodruff (Ed.). Advances in computational and stochastic optimization, logic programming, and heuristic search. Kluwer Academic Publishers, Dordrecht, 97–148, 1998.

[3] M. Libura. Sensitivity analysis for minimum hamiltonian path and traveling salesman problems. Discrete Applied Mathematics **30** 197–211, 1991.

[4] R. Ramasvamy, N. Chakravarti. Complexity of determining exact tolerances for min-sum and min-max combinatorial optimization problems. Working Paper WPS-247/95, Indian Institute of Management, Calcutta, India, 1995, pp. 34.

[5] S. van Hoesel, A. Wagelmans. On the complexity of postoptimality analysis of 0/1 programs. Discrete Applied Mathematics **91** 251–263, 1999.

[6] E.S. van der Poort, M. Libura, G. Sierksma, J.A.A. van der Veen. Solving the k-best traveling salesman problem. Computers & Operations Research **26** 409–425, 1999.